Linear Difference Equation

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1 First-Order Difference Equation

A linear difference equation can be defined as an equation that relates a variable y_t to its previous values linearly. The simplest one is a first-order scalar linear difference equation such as,

$$y_t = \lambda y_{t-1} + bx_t + a, \forall t, \tag{1}$$

where x_t is an exogenous variable and a is an constant. This is a first-order difference equation, since only first lag of its own values y_{t-1} appears in the equation. Here, we are interested in finding a solution of y_t in terms of current, past, or future values of exogenous variable x_t and (less importantly) a constant a. Put it different, we want to characterize the sequence $\{y_t\}$ in terms of $\{x_t\}$.

Using lag operator, we can rewrite (1) as follows.

$$(1 - \lambda L)y_t = bx_t + a \tag{2}$$

Dividing both sides by $(1 - \lambda L)$, we can obtain the particular solution for (1) as follows.

$$\hat{y}_t = \frac{bx_t}{1 - \lambda L} + \frac{a}{1 - \lambda} \tag{3}$$

Note that since a is a constant, $\frac{a}{1-\lambda L} = \frac{a}{1-\lambda}$ irrespective of the size of $|\lambda|$. In order to get the general solution, we need to add a term to (3). For this purpose, let's suppose that $\tilde{y} = \hat{y} + \omega_t$ is also a solution to (2). Plugging $\tilde{y}$ into (2), we get

$$(1 - \lambda L)\tilde{y}_t = (1 - \lambda L)\hat{y}_t + \omega_t - \lambda\omega_{t-1} = bx_t + a + \omega_t - \lambda\omega_{t-1}$$

Therefore, as long as $\omega_t = \lambda\omega_{t-1}$, $\tilde{y}_t$ is also a solution. Note that $\omega_t = \lambda\omega_{t-1} = \cdots = \lambda^t\omega_0$, for arbitrary initial value $\omega_0 = c$. Hence, the general solution is,

$$y_t = \frac{bx_t}{1 - \lambda L} + \frac{a}{1 - \lambda} + \lambda^t c, \tag{4}$$

where c is an arbitrary constant.

Let's suppose that $\{x_t\}$ is a bounded sequence, and we are interest in finding the bounded solution $\{y_t\}$. Then, when $|\lambda| < 1$, the bounded sequence $\{y_t\}$ from (4) can be obtained by following backward representation with $c = 0$.

$$\begin{aligned} y_t &= b\left(1 + \lambda L + (\lambda L)^2 + \cdots\right) x_t + \frac{a}{1 - \lambda} \\ &= b\sum_{j=0}^{\infty} \lambda^j x_{t-j} + \frac{a}{1 - \lambda} \end{aligned} \tag{5}$$

If $|\lambda| > 1$, we need to use forward representation in order to get the bounded sequence $\{y_t\}$ as follows.

$$\begin{aligned} y_t &= \frac{-(\lambda L)^{-1}}{1 - (\lambda L)^{-1}} bx_t + \frac{a}{1 - \lambda} \\ &= -b\sum_{j=1}^{\infty} \lambda^{-j} x_{t+j} + \frac{a}{1 - \lambda}, \end{aligned} \tag{6}$$

again, setting $c = 0$. Note that if we don't set c to 0, the sequence $\{y_t\}$ will diverge, that is, either $\lim_{t\to\infty} |y_t| = \infty$ or $\lim_{t\to-\infty} |y_t| = \infty$.

2 Second-Order Difference Equation

A second-order difference equation involves two lags of y_t terms in a linear difference equation. Let's consider following second-order difference equation.

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + bx_t + a, \forall t \tag{7}$$

Using lag operator, we can rewrite (7) as follows.

$$(1 - \phi_1 L - \phi_2 L^2) y_t = bx_t + a$$

or

$$(1 - \lambda_1 L)(1 - \lambda_2 L) y_t = bx_t + a,$$

where $\lambda_1 \lambda_2 = -\phi_2$ and $\lambda_1 + \lambda_2 = \phi_1$. Note that the characteristic roots from the characteristic equation, $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$, are $z_1 = 1/\lambda_1$ and $z_2 = 1/\lambda_2$.

Assuming $\lambda_1 \neq \lambda_2$ and $\lambda_i \neq 1, \forall i$, it turns out that the general solution to (7) is,

$$y_t = \frac{1}{1 - \lambda_1 L} \frac{1}{1 - \lambda_2 L} bx_t + \frac{1}{1 - \lambda_1} \frac{1}{1 - \lambda_2} a + \lambda_1^t c_1 + \lambda_2^t c_2 \tag{8}$$

It can also be shown that if $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_i \neq 1, \forall i$, the general solution would be,

$$y_t = \frac{bx_t}{(1 - \lambda L)^2} + \frac{a}{(1 - \lambda L)^2} + \lambda^t c_1 + t\lambda^t c_2 \tag{9}$$

Again, if we are interested in a bounded sequence of $\{y_t\}$ mapped from a bounded sequence $\{x_t\}$, then we need to set both c's to zero, and focus on a particular solution.

2.1 Distinct Real Eigenvalues

If all eigenvalues are distinct as in (8), then following is valid.

$$\frac{1}{1 - \lambda_1 L} \frac{1}{1 - \lambda_2 L} = \frac{1}{\lambda_1 - \lambda_2} \left(\frac{\lambda_1}{1 - \lambda_1 L} - \frac{\lambda_2}{1 - \lambda_2 L} \right) \tag{10}$$

If norms of both eigenvalues are strictly less than unity, that is, if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, we can solve (10) backward so that

$$\begin{aligned} \frac{1}{1 - \lambda_1 L} \frac{1}{1 - \lambda_2 L} &= \frac{1}{\lambda_1 - \lambda_2} \left(\lambda_1 \sum_{j=0}^{\infty} (\lambda_1 L)^j - \lambda_2 \sum_{j=0}^{\infty} (\lambda_2 L)^j \right) \\ &= \frac{1}{\lambda_1 - \lambda_2} \sum_{j=0}^{\infty} \left(\lambda_1^{j+1} - \lambda_2^{j+1} \right) L^j \end{aligned} \tag{11}$$

If, without loss of generality, $|\lambda_1| < 1$ and $|\lambda_2| > 1$, then

$$\begin{aligned} \frac{1}{1 - \lambda_1 L} \frac{1}{1 - \lambda_2 L} &= \frac{1}{\lambda_1 - \lambda_2} \left(\frac{\lambda_1}{1 - \lambda_1 L} + \frac{L^{-1}}{1 - (\lambda_2 L)^{-1}} \right) \\ &= \frac{\lambda_1}{\lambda_1 - \lambda_2} \sum_{j=0}^{\infty} (\lambda_1 L)^j + \frac{\lambda_2}{\lambda_1 - \lambda_2} \sum_{j=1}^{\infty} (\lambda_2 L)^{-j} \end{aligned} \tag{12}$$

Lastly, if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then

$$\begin{aligned} \frac{1}{1 - \lambda_1 L} \frac{1}{1 - \lambda_2 L} &= \frac{1}{\lambda_1 - \lambda_2} \left(\frac{-L^{-1}}{1 - (\lambda_1 L)^{-1}} + \frac{L^{-1}}{1 - (\lambda_2 L)^{-1}} \right) \\ &= -\frac{1}{\lambda_1 - \lambda_2} \sum_{j=0}^{\infty} \left(\lambda_1^{-(j+1)} - \lambda_2^{-(j+1)} \right) L^{-(j+2)} \end{aligned} \tag{13}$$

Plugging these solutions to (8) for each case, we get the solutions for y_t.

2.2 Repeated Real Eigenvalues

When $\lambda_1 = \lambda_2 = \lambda$ and $|\lambda| < 1$, we can show that

$$\frac{1}{1-\lambda_1 L}\frac{1}{1-\lambda_2 L} = \frac{1}{(1-\lambda L)^2} = \sum_{j=0}^{\infty}(j+1)(\lambda L)^j, \tag{14}$$

while if $|\lambda| > 1$,

$$\frac{1}{1-\lambda_1 L}\frac{1}{1-\lambda_2 L} = \frac{1}{(1-\lambda L)^2} = \sum_{j=0}^{\infty}(j+1)(\lambda L)^{-(j+2)} \tag{15}$$

Plugging these solutions to (9) for each case, we get the solutions for y_t.

2.3 Complex Eigenvalues

It should be noted that if one eigenvalue turns out to be a complex number, then the other eigenvalue is the complex conjugate of it, that is, $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$, where $\lambda_1 + \lambda_2 = 2\alpha = \phi_1$, $\lambda_1\lambda_2 = \alpha^2 + \beta^2 = -\phi_2$, and $|\lambda_i| = \sqrt{\alpha^2+\beta^2}$, $\forall i$. Using useful polar representation, $\lambda_1 = re^{iw} = r(\cos w + i \sin w)$, $\lambda_2 = re^{-iw} = r(\cos w - i \sin w)$, where $r = \sqrt{\alpha^2+\beta^2}$ and $\tan w = \frac{\beta}{\alpha}$.

Note that norms of complex eigenvalues are same. If $|\lambda| < 1$, from (11),

$$\begin{aligned}\frac{1}{1-\lambda_1 L}\frac{1}{1-\lambda_2 L} &= \frac{1}{\lambda_1-\lambda_2}\sum_{j=0}^{\infty}\left(\lambda_1^{j+1}-\lambda_2^{j+1}\right)L^j \\ &= \frac{1}{\sin w}\sum_{j=0}^{\infty} r^j \sin\left(w(j+1)\right)L^j \end{aligned} \tag{16}$$

If $|\lambda| > 1$, from (13),

$$\begin{aligned}\frac{1}{1-\lambda_1 L}\frac{1}{1-\lambda_2 L} &= -\frac{1}{\lambda_1-\lambda_2}\sum_{j=0}^{\infty}\left(\lambda_1^{-(j+1)}-\lambda_2^{-(j+1)}\right)L^{-(j+2)} \\ &= -\frac{1}{\sin w}\sum_{j=0}^{\infty} r^{-(j+2)} \sin\left(w(j+1)\right)L^{-(j+2)} \end{aligned} \tag{17}$$

Plugging these solutions to (8) for each case, we obtain the solutions for y_t.

3 Vector Difference Equation

Let's consider following first order vector difference equation.

$$\mathbf{y}_t = \mathbf{F}\mathbf{y}_{t-1} + \mathbf{B}\mathbf{x}_t + \mathbf{a}, \ \forall t, \tag{18}$$

where $\mathbf{y}_t$ is a $k \times k$ vector, $\mathbf{F}$ is a $k \times k$ matrix, $\mathbf{B}$ is a $k \times m$ matrix, $\mathbf{x}_t$ is a $m \times 1$ vector, and $\mathbf{a}$ is a $k \times 1$ constant vector.

3.1 Diagonalizable F

If $\mathbf{F}$ is diagonalizable so as that $\mathbf{F} = \mathbf{T\Lambda T}^{-1}$ exists where $\mathbf{\Lambda}$ is a $k \times k$ diagonal matrix with the eigenvalues (λ_i, $i = 1, \cdots, k$) on the diagonal, and $\mathbf{T}$ is a $k \times k$ matrix whose columns are corresponding eigenvectors that are arbitrarily scaled. Then, premultiplying $\mathbf{T}^{-1}$ to (18), and redefining $\mathbf{z}_t = \mathbf{T}^{-1}\mathbf{y}_t$, we get,

$$\mathbf{z}_t = \mathbf{\Lambda}\mathbf{z}_{t-1} + \mathbf{T}^{-1}\mathbf{B}\mathbf{x}_t + \mathbf{T}^{-1}\mathbf{a} \tag{19}$$

Note that since $\mathbf{\Lambda}$ is diagonal, (19) can be equivalently described as following set of k scalar difference equations.

$$z_{t,i} = \lambda_i z_{t-1,i} + (\mathbf{T}^{-1}\mathbf{B})_i \mathbf{x}_t + (\mathbf{T}^{-1}\mathbf{a})_i \ , \ i = 1, \cdots, k, \tag{20}$$

where $(\mathbf{T}^{-1}\mathbf{B})_i$ and $(\mathbf{T}^{-1}\mathbf{a})_i$ are the i^{th} rows of $\mathbf{T}^{-1}\mathbf{B}$ and $\mathbf{T}^{-1}\mathbf{a}$, respectively.

Let's suppose that there are l eigenvalues that are greater than unity in norm, and $k-l$ eigenvalues that are less than unity. Arranging the eigenvalues in $\mathbf{\Lambda}$ from the smallest to the largest in norm, we can rewrite the system as following partitioned matrices.

$$\mathbf{y}_t = \begin{pmatrix} \mathbf{y}_{1t} \\ \mathbf{y}_{2t} \end{pmatrix} \ \mathbf{F} = \begin{pmatrix} \mathbf{F}_{11} & \mathbf{F}_{12} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{pmatrix} \ \mathbf{B}_t = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{pmatrix} \ \mathbf{a}_t = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{pmatrix} \ \mathbf{\Lambda} = \begin{pmatrix} \mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2 \end{pmatrix} \ \mathbf{T}^{-1} = \begin{pmatrix} \mathbf{T}^{11} & \mathbf{T}^{12} \\ \mathbf{T}^{21} & \mathbf{T}^{22} \end{pmatrix} \ \mathbf{z}_t = \mathbf{T}^{-1}\mathbf{y}_t = \begin{pmatrix} \mathbf{z}_{1t} \\ \mathbf{z}_{2t} \end{pmatrix},$$

where $\mathbf{y}_{1t}$, $\mathbf{a}_1$, and $\mathbf{z}_{1t}$ are $(k-l)\times 1$ vectors, $\mathbf{F}_{11}$, $\mathbf{T}_{11}$, $\mathbf{\Lambda}_1$, $\mathbf{T}^{11}$ are $(k-l)\times(k-l)$ matrices, and $\mathbf{B}_1$ is $(k-l)\times m$. Then, the first $(k-l)$ equations can be written as,

$$\mathbf{z}_{1,t} = \mathbf{\Lambda}_1 \mathbf{z}_{1,t-1} + (\mathbf{T}^{11}\mathbf{B}_1 + \mathbf{T}^{12}\mathbf{B}_2)\mathbf{x}_t + (\mathbf{T}^{11}\mathbf{a}_1 + \mathbf{T}^{12}\mathbf{a}_2), \tag{21}$$

and the remaining l equations are,

$$\mathbf{z}_{2,t} = \mathbf{\Lambda}_2 \mathbf{z}_{2,t-1} + (\mathbf{T}^{21}\mathbf{B}_1 + \mathbf{T}^{22}\mathbf{B}_2)\mathbf{x}_t + (\mathbf{T}^{21}\mathbf{a}_1 + \mathbf{T}^{22}\mathbf{a}_2) \tag{22}$$

Since first set of the equations in (21) has the roots less than the unity in norm, we can solve it backward, whereas equations in (22) can be solved forward yielding following particular solutions.

$$\mathbf{z}_{1,t} = \sum_{j=0}^{\infty} \mathbf{\Lambda}_1^j (\mathbf{T}^{11}\mathbf{B}_1 + \mathbf{T}^{12}\mathbf{B}_2)\mathbf{x}_{t-j} + (\mathbf{I} - \mathbf{\Lambda}_1)^{-1}(\mathbf{T}^{11}\mathbf{a}_1 + \mathbf{T}^{12}\mathbf{a}_2) \tag{23}$$

$$\mathbf{z}_{2,t} = -\sum_{j=0}^{\infty} \mathbf{\Lambda}_2^{-(j+1)} (\mathbf{T}^{21}\mathbf{B}_1 + \mathbf{T}^{22}\mathbf{B}_2)\mathbf{x}_{t+1+j} - (\mathbf{I} - \mathbf{\Lambda}_2^{-1})^{-1}\mathbf{\Lambda}_2^{-1}(\mathbf{T}^{21}\mathbf{a}_1 + \mathbf{T}^{22}\mathbf{a}_2) \tag{24}$$

For general solutions, we need to add $c_i\lambda_i^t$, $i = 1, \cdots, k$, for each equation, where c_i's are any constants.. However, since we are interested only in bounded sequences, we will ignore these terms by setting $c_i = 0$, $\forall i$.

Once we obtain solutions by (23) and (24), the solutions for the original variables in $\mathbf{y}_t$ in (18) can be recovered by $\mathbf{y}_t = \mathbf{T}\mathbf{z}_t$. That is,

$$\begin{pmatrix} \mathbf{y}_{1t} \\ \mathbf{y}_{2t} \end{pmatrix} = \begin{pmatrix} \mathbf{T}_{11}\mathbf{z}_{1,t} + \mathbf{T}_{12}\mathbf{z}_{2,t} \\ \mathbf{T}_{21}\mathbf{z}_{1,t} + \mathbf{T}_{22}\mathbf{z}_{2,t} \end{pmatrix} \tag{25}$$

In general, all variables in $\mathbf{y}_t$ would have both forward and backward looking components, since $\mathbf{z}_{1,t}$ has backward looking solutions, while $\mathbf{z}_{2,t}$ has forward looking solutions.

3.2 Undiagonalizable F

Any undiagonalizable matrix turns out to have repeated eigenvalues although not all matrix that have repeated eigenvalues are not diagonalizable. The identity matrix or any multiple of it is a counter example. Even when $\mathbf{F}$ in (18) is not diagonalizable, the Jordan decomposition still can apply so that $\mathbf{F} = \mathbf{M}\mathbf{J}\mathbf{M}^{-1}$ exists, where $\mathbf{M}$ is a $(k \times k)$ matrix and $\mathbf{J}$ is the matrix that has following form.

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_s \end{pmatrix}, \tag{26}$$

where $s < k$ is the number of distinct eigenvalues and $\mathbf{J}_i$ is a $(n_i \times n_i)$ matrix with following form where n_i is the number of times that the eigenvalue λ_i is repeated.

$$\mathbf{J}_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}, \tag{27}$$

where $i = 1, 2, \cdots, s$.

Redefining $\mathbf{z}_t = \mathbf{M}^{-1}\mathbf{y}_t$, we can rewrite (18) as follows.

$$\mathbf{z}_t = \mathbf{J}\mathbf{z}_{t-1} + \mathbf{M}^{-1}\mathbf{B}\mathbf{x}_t + \mathbf{M}^{-1}\mathbf{a} \tag{28}$$

The system of equations in (28) can be easily solved recursively in following ways. Let's suppose $n_1 = 3,\ n_2 = 5, \cdots,\ n_s = k - \sum_{i=1}^{s-1} n_i$. Note that z_{3t}, the last row variable in $\mathbf{J}_1$, satisfies $z_{3t} = \lambda_1 z_{3t-1} + \left(\mathbf{M}^{-1}\mathbf{B}\right)_3 \mathbf{x}_t + \left(\mathbf{M}^{-1}\mathbf{a}\right)_3$, where $\left(\mathbf{M}^{-1}\mathbf{B}\right)_3$ and $\left(\mathbf{M}^{-1}\mathbf{a}\right)_3$ are the third rows of $\mathbf{M}^{-1}\mathbf{B}$ and $\mathbf{M}^{-1}\mathbf{a}$, respectively. With z_{3t} at hand, z_{2t} can be solved by $z_{2t} = \lambda_1 z_{2t-1} + z_{3t} + \left(\mathbf{M}^{-1}\mathbf{B}\right)_2 \mathbf{x}_t + \left(\mathbf{M}^{-1}\mathbf{a}\right)_2$. Of course, z_{1t} satisfies $z_{1t} = \lambda_1 z_{1t-1} + z_{2t} + \left(\mathbf{M}^{-1}\mathbf{B}\right)_1 \mathbf{x}_t + \left(\mathbf{M}^{-1}\mathbf{a}\right)_1$. Same procedure can be done for all other variables in $\mathbf{J}_2$ to $\mathbf{J}_s$ rows. Finally, once we solve for all variables by this way, the original series can be restored by

$$\mathbf{y}_t = \mathbf{M}\mathbf{z}_t \tag{29}$$

3.3 p^{th} Order Scalar Difference Equation

Let's consider following p^{th} order scalar difference equation.

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + b x_t + a,\ \forall t \tag{30}$$

Such a scalar difference equation can be represented by a similar vector difference equation as in (18),

$$\mathbf{y}_t = \mathbf{F}\mathbf{y}_{t-1} + \mathbf{b}x_t + \mathbf{a},\ \forall t, \tag{31}$$

where x_t is a scalar, and

$$\underset{(p\times 1)}{\mathbf{y}_t} = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix}, \quad \underset{(p\times p)}{\mathbf{F}} = \begin{pmatrix} \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_p \end{bmatrix} \\ \underset{(p-1)\times(p-1)}{\mathbf{I}} \;\vdots\; \underset{(p-1)\times 1}{\mathbf{0}} \end{pmatrix} \tag{32}$$

$$\underset{(p\times 1)}{\mathbf{y}_{t-1}} = \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p} \end{pmatrix}, \quad \underset{(p\times 1)}{\mathbf{b}} = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \underset{(p\times 1)}{\mathbf{a}} = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This type of representation is also called *State Space Representation*.

In order to solve this system of equations by methods described before, we need to obtain the eigenvalues of the matrix $\mathbf{F}$. Note that the eigenvalues are the solutions from $|\mathbf{F} - \lambda\mathbf{I_p}| = \mathbf{0}_{(p\times p)}$. It is easy to show that the eigenvalues λ's of $\mathbf{F}$ satisfy following characteristic equation.

$$\lambda^p - \phi_1\lambda^{p-1} - \phi_2\lambda^{p-2} - \cdots - \phi_{p-1}\lambda - \phi_p = 0 \tag{33}$$

Or we can solve following equivalent p^{th} order polynomial equation.

$$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0, \tag{34}$$

where the solutions to (34) z's are the inverses of λ's. Once we obtain the eigenvalues, it is straightforward but tedious to solve the system by aforementioned methods.