Structural Vector Autoregressive Process

Hyeongwoo Kim

Auburn University

April 2008

1 SVAR Model

Consider the following *reduced form* $k-$variate vector autoregressive (VAR) process.

$$\mathbf{y}_t = \mathbf{B}_1\mathbf{y}_{t-1} + \mathbf{B}_2\mathbf{y}_{t-2} + \cdots + \mathbf{B}_p\mathbf{y}_{t-p} + \varepsilon_t \tag{1}$$

where $\mathbf{y}_t = [y_{1,t}\ y_{2,t}\ \cdots\ y_{k,t}]'$, $\varepsilon_t = [\varepsilon_{1,t}\ \varepsilon_{2,t}\ \cdots\ \varepsilon_{k,t}]'$, and $\mathbf{B}$s are $k \times k$ matrices. Using lag polynomial, (1) can be rewritten as,

$$\mathbf{B}(L)\mathbf{y}_t = \varepsilon_t, \tag{2}$$

where $\mathbf{B}(L) = \mathbf{I} - \mathbf{B}_1 L - \cdots - \mathbf{B}_p L^p$.

Note that the variance-covariance matrix can be obtained by

$$E\varepsilon_t\varepsilon_t' = \mathbf{\Sigma}, \tag{3}$$

which is a *symmetric* and *positive semidefinite* $k \times k$ matrix. In general, the off-diagonal elements of $\mathbf{\Sigma}$ will be non-zero values, which means that the noise terms in ε_t are mutually correlated. This feature may be problematic if one attempts to trace the dynamic responses (or adjustments) of the variables of interest to an economic shock such as inflation (demand) shocks and technology (supply) shocks.

We resolve this problem by constructing the following structural vector autoregressive (SVAR) process.

$$\mathbf{A}_0\mathbf{y}_t = \mathbf{A}_1\mathbf{y}_{t-1} + \mathbf{A}_2\mathbf{y}_{t-2} + \cdots + \mathbf{A}_p\mathbf{y}_{t-p} + \mathbf{u}_t, \tag{4}$$

where

$$E\mathbf{u}_t\mathbf{u}_t' = \mathbf{I} \tag{5}$$

and $\mathbf{A}_0$ is a $k \times k$ matrix that describes the contemporaneous relations between the k variables in this model. Note that we assume that the terms in $\mathbf{u}_t$ are mutually orthogonal and have unit variances.[1] Note that, unlike the reduced form VAR in (2), the assumption in (5) enables us to make meaningful economic interpretation of the economy. Using lag polynomial, we have the following alternative representation.

$$\mathbf{A}(L)\mathbf{y}_t = \mathbf{u}_t, \tag{6}$$

where $\mathbf{A}(L) = \mathbf{A}_0 - \mathbf{A}_1 L - \cdots - \mathbf{A}_p L^p$. By pre-multiplying $\mathbf{A}_0^{-1}$ to both sides of (4), we obtain,

$$\mathbf{y}_t = \mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{y}_{t-1} + \cdots + \mathbf{A}_0^{-1}\mathbf{A}_p\mathbf{y}_{t-p} + \mathbf{A}_0^{-1}\mathbf{u}_t, \tag{7}$$

where

$$\mathbf{A}_0^{-1}\mathbf{u}_t = \varepsilon_t,\ \mathbf{A}(L) = \mathbf{A}_0\mathbf{B}(L),\ \mathbf{A}_j = \mathbf{A}_0\mathbf{B}_j,\ j = 1, 2, \cdots, p. \tag{8}$$

One can estimate $\mathbf{B}$s and Σ by the equation-by-equation least squares method. However, in order to fully describe the structural model (4), we need to identify $\mathbf{A}_0$. It is easy to see one can obtain $k(k+1)/2$ information from $\mathbf{\Sigma}$, which is a symmetric matrix. The problem is that we have k^2 unknowns in $\mathbf{A}_0$. This means that we need to impose $k(k-1)/2$ identifying assumptions. For example, if we have a bivariate ($k = 1$) SVAR model, one needs to impose one additional assumption, perhaps from economic theories or common beliefs. We discuss two popular ways to (just) identify the SVAR system in the following two sections.[2]

2 Short-Run Assumptions (Sims 1980)

Sims (1980) propose an identification method that relies on the Choleski decomposition of $\mathbf{\Sigma}$ (recursive method).

For simplicity, consider a bivariate SVAR model of the aggregate supply and the aggregate demand

[1] One can always transform the model to have unit variances by normalizing the system of equations.

[2] If we impose more than $k(k-1)/2$ assumptions, the system is over-identified. In this case, one can implement a specification test for the system. If we have less than $k(k-1)/2$ assumptions, the system is not identified.

equations, where $y_{1,t}$ is the real GDP growth rate (Δy_t) and $y_{2,t}$ is the GDP-deflator inflation rate (Δp_t). It may be reasonable to assume that the demand shock does not have any impact on $y_{1,t}$ contemporaneously. For instance, when the Fed implements an unanticipated interest rate cut, it may affect the real sector with a time lag. This implies that $\mathbf{A}_0$ is a lower-triangular matrix, so is $\mathbf{A}_0^{-1}$.

From $\mathbf{A}_0^{-1}\mathbf{u}_t = \varepsilon_t$, we obtain the following.

$$\mathbf{A}_0^{-1}\mathbf{A}_0^{-1\prime} = \mathbf{\Sigma} \tag{9}$$

By the Choleski decomposition, we get

$$\mathbf{\Sigma} = \mathbf{P}\mathbf{P}' \tag{10}$$

It is know that the Choleski decomposition of a symmetric and positive semidefinite matrix is unique. It follows then,

$$\mathbf{P}^{-1} = \mathbf{A}_0 \tag{11}$$

Since we identified $\mathbf{A}_0$, other structural coefficients $\mathbf{A}_j$ as well as the structural shock $\mathbf{u}_t$ can be identified by (8). Then, the impulse-response analysis can be implemented as follows.

For simplicity, consider the bivariate SVAR(1) model with the AD-AS equations. First, one needs to estimate the reduced form coefficients $\mathbf{B}$s and the variance-covariance matrix $\mathbf{\Sigma}$ by the least squares method. Then, obtain $\mathbf{P}$ by the Choleski decomposition of $\mathbf{\Sigma}$. By recursive substitution for the VAR(1) process,

$$\mathbf{y}_{t+j} = \mathbf{B}_1^{j+1}\mathbf{y}_{t-1} + \mathbf{P}\mathbf{u}_{t+j} + \mathbf{B}_1\mathbf{P}\mathbf{u}_{t+j-1} + \cdots + \mathbf{B}_1^{j}\mathbf{P}\mathbf{u}_t \tag{12}$$

That is, the impulse-response function of $\mathbf{y}_{t+j}$ is,

$$\psi_j = \mathbf{B}_1^{j}\mathbf{P}, \ j = 0, 1, \cdots \tag{13}$$

For a SVAR(p), one needs to use the *state-space representation* as usual. For the impulse-response function for the level variable, we need to obtain cumulative responses of the growth variable,

$$\zeta_j = \sum_{k=0}^{j}\mathbf{B}_1^{k}\mathbf{P}, \ j = 0, 1, \cdots \tag{14}$$

When one has more than 2 variables, she has to determine the ordering of the variables (Wold ordering). One drawback of Sims' method is that the impulse-response function may not be robust to the Wold ordering. Therefore, one may need to try many different orderings (robustness check) unless she has very convincing theories that justify a specific ordering. Alternatively, one may use the generalized impulse-response analysis proposed by Pesaran and Shin (1998) that is ordering-free.

3 Long-Run Assumptions (Blanchard and Quah 1989)

Blanchard and Quah proposed an identifying scheme that relies on the long-run proposition in economics. For example, the *classical dichotomy* or *money neutrality* are long-run propositions that no nominal shock has a permanent impact on real variables.

Consider the bivariate SVAR model of AD-AS equations again. From (6), we obtain the following moving average representation.[3]

$$\mathbf{y}_t = \mathbf{A}(L)^{-1}\mathbf{u}_t = \mathbf{C}(L)\mathbf{u}_t, \tag{15}$$

where $\mathbf{C}(L) = \mathbf{C}_0 + \mathbf{C}_1 L + \mathbf{C}_2 L^2 + \cdots$. Then, the long-run effects on the *level* variables are,

$$\sum_{j=0}^{\infty} \mathbf{C}_j = \mathbf{C}(1) \tag{16}$$

Note that the classical dichotomy implies that $\mathbf{C}(1)$ is a lower-triangular matrix and so is $\mathbf{A}(1)$. From the reduced form model,

$$\mathbf{y}_t = \mathbf{B}(L)^{-1}\varepsilon_t = \mathbf{B}(L)^{-1}\mathbf{A}_0^{-1}\mathbf{u}_t = (\mathbf{A}_0\mathbf{B}(L))^{-1}\mathbf{u}_t \tag{17}$$

Comparing (15) with (17),

$$\mathbf{C}(1) = (\mathbf{A}_0\mathbf{B}(1))^{-1} \tag{18}$$

Therefore, $\mathbf{A}_0\mathbf{B}(1)$ must be lower-triangular too. From (17) and (18),

$$\mathbf{B}(1)^{-1}\mathbf{\Sigma}\mathbf{B}(1)^{-1\prime} = (\mathbf{A}_0\mathbf{B}(1))^{-1}(\mathbf{A}_0\mathbf{B}(1))^{-1\prime} = \mathbf{C}(1)\mathbf{C}(1)' \tag{19}$$

[3] If y_t and p_t are nonstationary processes, their growth rates would be stationary processes so that $\mathbf{A}(L)$ is invertible.

Hence, by the Choleski decomposition of $\mathbf{B}(1)^{-1}\boldsymbol{\Sigma}\mathbf{B}(1)^{-1\prime}$ from the least squares estimation, we just identified the system ($\mathbf{A}_0$ identified), and obtain the long-run effects on the level variables. The effects on the growth variables can be similarly obtained. The impulse-response analysis can be also implemented as in previous section.